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Control Theory and Dynamical Systems

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Introduction

The goal of control theory (or mathematical systems theory) is to influence dynamical systems such that they achieve a desired behavior, e.g. stability. This requires the analysis of the system behavior under admissible control actions and, possibly, under perturbations.

The main mathematical areas involved are
- ordinary/partial differential equations (possibly including algebraic constraints), discrete-time systems
- random systems
- optimization/optimal control
- numerics
- basic mathematics like linear algebra, complex and functional analysis, differential geometry, ...

Application areas

J. C. Maxwell, On governors, Proc. Royal Soc. (1868).

The origins and the main applications are in engineering, in particular,

- mechanics,
- electromagnetic and electrical systems
- heat transfer

Many other applications, e.g., in economics, biology, ...

Recent trends:

- cyber-physical systems
- (digitally) networked control systems

Here: deterministic finite dimensional systems

We consider

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)),\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in \Omega \text{ for } t \in \mathbb{R}\} \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

with control range $\Omega \subset \mathbb{R}^m$ and

$$\begin{aligned} x_{n+1} &= f(x_n, u_n),\ u = (u_n)_{n\in\mathbb{Z}} \in \{u : \mathbb{Z} \to \mathbb{R}^m \,|u_n \in \Omega \text{ for } n \in \mathbb{Z}\} \\ y_n &= h(x_n, u_n). \end{aligned}$$

Here $u(\cdot)$ is the input or control and $y(\cdot)$ is the output or observation. We assume that for every initial state $x(0) = x_0$ and every control function u there is a unique solution $\varphi(t, x_0, u), t \in \mathbb{R}$ ($\varphi(n, x_0, u), n \in \mathbb{Z}$, resp.).

Instability of the geostationary orbit of a satellite

Circular orbit at fixed altitude at 35600 km in the equator plane with the same velocity of rotation as the earth.
Influence the motion such that it returns to this orbit when it is slightly off.
Model (Trentelman et al.):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ [x_1(t)+R_0][x_4(t)+\omega]^2 - \frac{GM_E}{[x_1(t)+R_0]^2} + \frac{F_r(t)}{M_S} \\ x_4(t) \\ -\frac{2x_2(t)[x_4(t)+\omega]}{x_1(t)+R_0} + \frac{F_\theta(t)}{M_S[x_1(t)+R_0]} \end{bmatrix}$$

where R_0 radius of geostationary orbit,
x_1 deviation from R_0 and x_3 deviation from angle ω of earth rotation,
M_S, M_e mass of the satellite, earth
G earth's gravitational constant
$F_r(t), F_\theta(t)$ radial, angular force: **controls!** $y = x_3$ **Output.**
Equilibrium: $x_1 = x_2 = x_3 = x_4, F_r = 0, F_\theta = 0$, since $R_0\omega^2 - \frac{GM_E}{R_0^2} = 0$.

The stabilization problem

For initial states

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (x_1^0, x_2^0, x_3^0, x_4^0),$$

Kepler's law implies that the resulting orbit will be an ellipse with the earth in one of its focuses. The equilibrium position will not be asymptotically stable.

Problem: Find a feedback controller that generates a control input $u = (F_r, F_\theta)$ on the basis of the measured output $y = x_3$ such that the equilibrium position becomes locally asymptotically stable.

Linear control systems

Linear control systems have the form

$$\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t),$$

where A, B, C and D are matrices of appropriate dimension.
For the satellite model, linearize in the equilibrium. With $f : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4, g : \mathbb{R}^4 \to \mathbb{R}$, near $(0,0) \in \mathbb{R}^4 \times \mathbb{R}^2$

$$f(x,u) \approx f(0,0) + D_x f(0,0)x + D_u f(0,0)u.$$

Here $f(0,0) = 0$ and

$$D_x f(0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega R_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{R_0} & 0 & 0 \end{bmatrix}, D_u f(0,0) = \begin{bmatrix} 0 & 0 \\ \frac{1}{M_S} & 0 \\ 0 & 0 \\ 0 & \frac{1}{M_S R_0} \end{bmatrix}.$$

We obtain a linear control system with

$$A = D_x f(0,0), B = D_u f(0,0), C = [0,0,1,0], D = 0.$$

Stabilization

Local stabilization follows if the linearized system is stabilized.

For the linear system, use
- linear state feedback $u = Fx$, or
- a linear feedback controller

$$\begin{aligned} \dot{w}(t) &= Kw(t) + Ly(t) \\ u(t) &= Mw(t) + Ny(t) \end{aligned}$$

with appropriate matrices F, and K, L, M, N, resp.

Control-theoretic problems

The most fundamental dichotomy in problems and concepts concerns open-loop and closed-loop problems.

Open-loop problems are
- controllability
- observability
- optimal control

Closed-loop problems are
- stabilization
- tracking
- keeping the system in a “safe region”
- robustness against perturbations

2. Controllability and observability

Given a control system

$$\dot{x} = f(x, u),$$

is it possible to reach from any given initial state x_0 any final state x_1 using admissible control functions?
In particular, analyze this for linear control systems without and with control constraints.
For nonlinear systems, determine regions of complete controllability (control sets).

Given a measured output $y(t), t \in [0, T]$, determine the initial state $x(0)$ and the present state $x(t)$. When is this always possible (observability)?

3. Stabilization of linear and nonlinear systems

If the system should operate in a steady state, it should return to it under (small) deviations from the steady state. This can be based on knowledge of the state or, under observability, on measured outputs.

The linear theory is clear cut. In particular, linear-quadratic optimal control yields an efficient method to construct stabilizing controls.

For nonlinear problems there is a diversity of concepts, some of which will be discussed.

4. Control sets, the control flow and relations to random systems

Open-loop control systems can be viewed as skew product dynamical systems, if one includes the control functions (together with the time shift) into the state.
- topologically transitive sets $\leftrightarrow$ control sets
- chain transitivity $\leftrightarrow$ chain control sets

The term $u(t)$ can also be interpreted as a deterministic perturbation. Additionally, one may replace $u(t)$ by a stochastic perturbation. This leads to connections between control systems and degenerate Markov diffusions and piecewise deterministic Markov processes, resp.

5. Bilinear control systems

Bilinear systems have the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t), \ \ (u_i(t))_{i=1,\dots,m} \in \Omega \subset \mathbb{R}^m,$$

with matrices $A_0, A_1, \dots, A_m$ (obtained by linearization w.r.t. x only).

The associated control flow is a linear skew product flow with chain transitive base space.

Selgrade's theorem together with the Morse spectrum yields a spectral decomposition which gives information on stability and stabilization.

6. Invariance entropy

The classical feedback concept supposes that the values $x(t) \in \mathbb{R}^n$ are available for the controller. If this information is sent over a digital channel, e.g. over the internet, this concept is no longer available: The communication may be constrained by finite bit rates, loss of information packages, delays etc.

I will concentrate on the problem of finite bit rates, where the concept of invariance entropy has been developed in order to determine the information that is actually necessary for control tasks, here, making a subset of the state space invariant. This has close connections to topological and measure theoretic entropy of dynamical systems.

In particular, a data rate theorem will be presented.

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