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Stabilization of linear and nonlinear systems

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Introduction

Stabilization is one of the major themes in control theory. Very often, a primary goal is to ensure stability (or to improve stability properties), since otherwise the system may just explode.
Let us start with linear systems

$$\dot{x}(t) = Ax(t) + Bu(t),\ u(t) \in \mathbb{R}^m.$$

Controllability guarantees that one can reach $0 \in \mathbb{R}^d$ (in finite time) from each $x_0 \in \mathbb{R}^d$ by an appropriate control $u_{x_0}(\cdot)$.
However, if A has an eigenvalue with positive real part, the solution will diverge under arbitrarily small perturbations:

$$\varphi(t, x_0 + \varepsilon x_1, u_{x_0}) = \varepsilon \underbrace{e^{At}x_1}_{\to\infty \text{ generically}} + \underbrace{e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_{x_0}(s)ds}_{=0}.$$

Contents

- State feedbacks for stabilization: Pole-shifting theorem

- Stabilization via outputs: static output feedback, observers and dynamic output feedback

- Linear-quadratic optimal control

- Nonlinear stabilization:
Linearization, Brockett's necessary condition, Control-Lyapunov functions, Coron's return method, piecewise constant controls

State feedbacks

A remedy is to use feedbacks:

State feedback: Find a matrix F such that with $u = Fx$

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t).$$

is (asymptotically) stable.

Some observations:
(i) By coordinate transformation we may assume that

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \text{ with } (A_1, B_1) \text{ controllable.}$$

(ii) For scalar control and (A, b) controllable, we may assume

$$A = \begin{bmatrix} 0 & 1 & . & . & 0 \\ . & . & . & & . \\ . & . & & & 1 \\ \alpha_0 & \alpha_1 & . & . & \alpha_{n-1} \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \end{bmatrix}$$

with $\chi_A(z) = z^n - \alpha_{n-1} z^{n-1} - ... - \alpha_1 z - \alpha_0$.

(iii) This can be stabilized by

$$f = (\beta_0 - \alpha_0, \beta_1 - \alpha_1, \ldots, \beta_{n-1} - \alpha_{n-1}) \in \mathbb{R}^{1\times d},$$

since

$$A + bf = A + \begin{bmatrix} 0 \\ . \\ . \\ 1 \end{bmatrix} (\beta_0 - \alpha_0, \ldots, \beta_{n-1} - \alpha_{n-1}) = \begin{bmatrix} 0 & 1 & . & 0 \\ . & . & . & . \\ . & . & & . \\ \beta_0 & \beta_1 & . & \beta_{n-1} \end{bmatrix}.$$

with $\chi_A(z) = z^n - \beta_{n-1} z^{n-1} - ... - \beta_1 z - \beta_0$.

State feedbacks

(iv) (Heymann's Lemma) Let (A, B) be controllable and $b = Bv \neq 0$. Then there is F such that

$$(A + BF, b) \text{ is controllable.}$$

(ii) - (iv) imply that every controllable pair is stabilizable. Use (i) to get

Theorem. For (A, B) let χ be a normed polynomial χ with $\deg\chi = \dim \langle A \,|\, \mathrm{im} B \rangle$. Then there exists a feedback F s.t.

$$\chi_{A+BF} = \chi \cdot \chi_{A_3}.$$

This is known as the **pole shifting theorem**.

The theorem also shows that stabilizability is equivalent to asymptotic null controllability.

Laplace-transforms and poles

For initial condition $x(0) = 0$, take Laplace transforms

$$\hat{u}(s) = \int_0^\infty e^{-st}u(t)dt, \ \hat{x}(s) = \int_0^\infty e^{-st}x(t)dt.$$

By partial integration

$$\dot{x}\hat{}(s) = \int_0^\infty e^{-st}\dot{x}(t)dt = s\int_0^\infty e^{-st}x(t)dt = s\hat{x}(s).$$

Thus $\dot{x}\hat{}(s) = A\hat{x}(s) + B\hat{u}(s)$ implies

$$\hat{x}(s) = (sI - A)^{-1}B\hat{u}(s).$$

The eigenvalues of A are the poles of $(sI - A)^{-1}B$.

Stabilization via outputs

Consider $\dot{x} = Ax + Bu, \; y = Cx$.
Static output feedback: With $u = Fy = FCx$

$$\dot{x}(t) = Ax(t) + BFCx = (A + BFC)x(t).$$

Example: $\dot{x}_1 = x_2, \; \dot{x}_2 = u\,, \; y = x_1.$
This system is controllable and observable, but there is no (asymptotically) stabilizing feedback $k : \mathbb{R} \to \mathbb{R}$ with

$$\dot{x}_1 = x_2, \dot{x}_2 = k(y) = k(x_1).$$

In fact,

$$V(x_1, x_2) = (x_2)^2 - 2\int_0^{x_1} k(s)ds$$

is constant along trajectories, $V(0,0) = 0$ and $V(0,\alpha) = \alpha^2$ for $\alpha \neq 0$.
Thus static output feedback is not good enough !

Dynamic observers

Instead of this static output feedback use dynamic output feedback.

Separate the output stabilization problem into two subproblems:

(i) find a stabilizing state feedback;

(ii) estimate the state by a dynamical system, an observer, and use this estimate in (i).

A dynamic observer

ad (ii) For $\dot{x} = Ax + Bu, y = Cx$ find L such that $A + LC$ is stable.

Then, by linearity, the dynamic observer

$$\dot{z} = (A + LC)z - Ly + Bu$$

satisfies

$$\|z(t) - x(t)\| \longrightarrow 0 \text{ for } t \longrightarrow \infty.$$

In fact: the error $e(t) = z(t) - x(t)$ converges to 0, since

$$\begin{aligned} \dot{e} &= \dot{z} - \dot{x} = (A + LC)z - Ly + Bu - Ax - Bu \\ &= (A + LC)z - LCx - Ax \\ &= (A + LC)(z - x) \\ &= (A + LC)e. \end{aligned}$$

Compensator

Theorem. If (A, B) and $(A^\top, C^\top)$ are stabilizable (i.e., asymptotic null controllability and asymptotic observability hold), then there are F and L such that following the dynamic output feedback stabilizes the system,

$$u = Fz,$$

where

$$\dot{z} = (A + LC)z + BFz - LCx.$$

Compensator: comments on the proof

We use the estimate $z(t)$ instead of the state $x(t)$ in the state feedback and assume that (A, B) and $(A^\top, C^\top)$ are stabilizable.

Then the system is stabilized by $u = Fz$, since the following coupled system is stable,

$$\begin{aligned} \dot{x} &= Ax + BFz \\ \dot{z} &= (A + LC)z + BFz - LCx. \end{aligned}$$

In fact, one can prove stability for the corresponding system matrix

$$\begin{bmatrix} A & BF \\ -LC & A + LC + BF \end{bmatrix}.$$

Linear-quadratic optimal control

This is an efficient (and intensely studied) method to construct stabilizing feedbacks. Consider

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) + Du(t). \end{aligned}$$

Here $z(t)$ is the output which is to be controlled. This can be done by minimizing for given initial state x_0 over u

$$J(x_0; u) = \int_0^\infty \left[\|Cx(t)\|^2 + \|Du(t)\|^2 \right] dt.$$

More generally, minimize with $Q \geq 0$ and $N > 0$,

$$J(x_0; u) = \int_0^\infty \left[x(t)^\top Q x(t) + u(t)^\top N u(t) \right] dt.$$

For $Q > 0$, $x(t) \to 0$ for $t \to \infty$ if there is u with $J(x_0; u) < \infty$.

Goal: Show that the optimal controls can be written as feedback $u = Fx$.

This problem is closely related to positive semidefinite solutions of the algebraic matrix Riccati equation

$$A^\top P + PA - PBB^\top P + Q = 0. \qquad \text{(ARE)}$$

A typical result:

Theorem. Assume that (A, B) is stabilizable and $\mathrm{spec}(A) \cap \imath\mathbb{R} = \varnothing$.

(i) There is a smallest positive semidefinit solution P^- of ARE.

(ii) For every input u

$$J(x_0; u) = x_0^\top P^- x_0 + \int_0^\infty \left\| u(t) + B^\top P^- x(t) \right\|^2 dt.$$

(iii) The optimal input is given by the feedback

$$u(t) = -B^\top P^- x(t).$$

The **proof** uses the finite time problem and completion of squares.

An example

Stabilize an inverted pendulum on a flying quadrocopter.

The complete system is described by a 16-dimensional system of differential equations (12 for the quadrocopter + 4 for the pendulum) with 4 control inputs.
After simplification to 13 dimensions and linearization in the equilibrium a linear-quadratic optimal control problem is solved.
Critical is the measurement of the states which is done by an infrared motion tracking system.

Hehn and D'Andrea, IEEE Trans. Aut. Control (2011)

Further problems

The H^∞-problem for

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ z &= Cx + Du \end{aligned}$$

Goal: Given $\gamma > 0$ find F such that $A + BF$ is stable and (for $x_0 = 0$)

$$\|z\|_2 \leq \gamma \|d\|_2 \text{ for all perturbations } d \in L^2(0, \infty, \mathbb{R}^\ell).$$

This is possible for $\gamma > \|G_F\|$ with

$$G_F : L^2(0, \infty) \to L^2(0, \infty), d(\cdot) \mapsto z(\cdot) = \int_0^{\cdot} Ce^{(A+BF)(t-\tau)} Ed(\tau) \, \mathrm{d}\tau.$$

(well defined for $A + BF$ stable)
This again leads to LQ-optimal control (without positive definiteness).

Note that for stable A and

$$G : L^2(0,\infty) \to L^2(0,\infty), d(\cdot) \mapsto z(\cdot) = \int_0^{\cdot} Ce^{A(t-\tau)} Ed(\tau)\, \mathrm{d}\tau$$

and

$$\tilde{G}(s) = C(sI - A)^{-1} E$$

one has

$$\|G\| = \sup \left\{ \frac{\|G(d)\|_2}{\|d\|_2} \,\Big|\, 0 \neq d \in L^2 \right\} = \sup_{\omega \in \mathbb{R}} \left\| \tilde{G}(i\omega) \right\|,$$

where $\|G(i\omega)\|$ denotes the largest singular value.
This is the H^∞-norm of matrix-valued functions which are holomorphic on the open right half plane.

Nonlinear stabilization at an equilibrium

Consider

$$\dot{x}(t) = f(x(t), u(t))$$

and let x^* be an equilibrium $f(x^*, u^*) = 0$. Linearization in (x^*, u^*) yields

$$\dot{y}(t) = f_x(x^*, u^*)y(t) + f_u(x^*, u^*)v(t)$$

and write $A = f_x(x^*, u^*)$ and $B = f_u(x^*, u^*)$.
Then a stabilizing feedback F for the linearized system is locally stabilizing for the nonlinear system

$$\dot{x}(t) = f(x(t), F(x(t) - x^*)).$$

(use a Lyapunov function)

Brockett's necessary condition

Theorem. Consider $\dot{x} = f(x, u)$, $u \in U$ open. If there is a locally stabilizing continuous feedback $F : \mathbb{R}^d \to U$, then $f(\mathbb{R}^d, U)$ is a neighborhood of 0.

Example (Brockett's nonholonomic integrator)

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

This is a simple model for a vehicle with angle $\theta = x_1$ in forward direction and position

$$(z_1, z_2) = (x_2 \cos\theta + x_3 \sin\theta, x_2 \sin\theta - x_3 \cos\theta).$$

No point $(0, 0, \varepsilon)$ with $\varepsilon \neq 0$ is in the image of f.
On the other hand, the system is asymptotically null controllable.

Control-Lyapunov functions

Asymptotic controllability to an equilibrium and stabilization can be dealt with using control-Lyapunov functions $V(x, u)$ which decrease along trajectories for appropriate controls.

Roughly,
- asymptotic controllability to an equilibrium holds if there exists a continuous control-Lyapunov function
- stabilizability with continuous feedback holds if there exists a smooth control-Lyapunov function.

cf. Sontag (1998), Coron (2007).

Coron's return method: time-varying feedbacks

Theorem. Consider a driftless control system in $\mathbb{R}^d$

$$\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x)$$

and assume that

$$\{g(x) \,|g \in \mathcal{LA}(f_1, \ldots, f_m)\} = \mathbb{R}^d \text{ for all } x \neq 0.$$

Then for every $T > 0$ there exists $u \in C^\infty(\mathbb{R}^d \times \mathbb{R})$ with

$$u(0, t) = 0, \;\; u(x, t + T) = u(x, t) \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}^d,$$

such that 0 is globally asymptotically stable for

$$\dot{x} = \sum_{i=1}^{m} u_i(x, t) f_i(x).$$

The proof constructs periodic trajectories near 0 with controllable linearization. Coron (1992), (2007).

Example

Nonholonomic integrator

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = x_1 u_2 - x_2 u_1.$$

Here

$$f_1(x) = \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix}, f_2(x) = \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}.$$

Brockett's necessary condition is violated, but the Lie algebra rank condition is satisfied. Hence it can be globally asymptotically stabilized by means of periodic time-varying feedback.

Stabilization with piecewise constant controls

Continuous stirred tank reactor

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1 - x_2)e^{x_1} \\ -x_2 + \alpha(1 - x_2)e^{x_1} \end{bmatrix} + u(t) \begin{bmatrix} x_c - x_1 \\ 0 \end{bmatrix},$$

where x_1 is the temperature and x_2 is the product concentration, x_c is the coolant temperature and the control affects the heat transfer coefficient with parameters

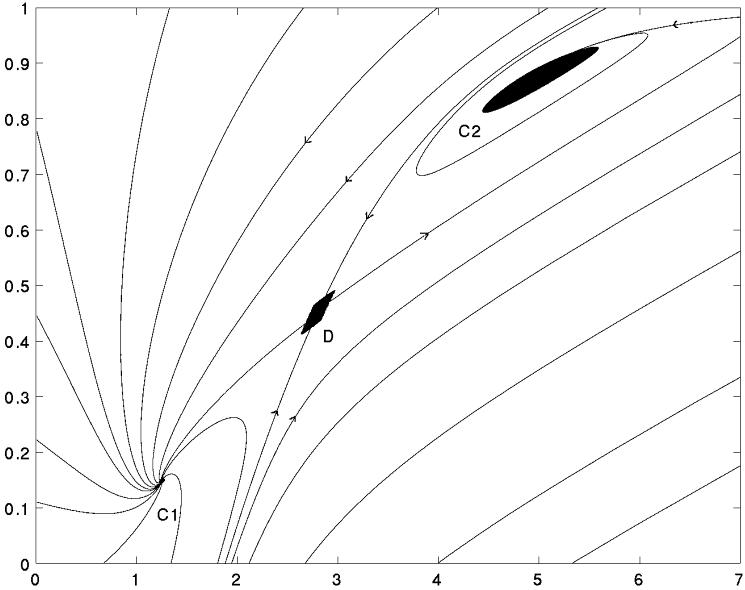
$$a = 0.95,\ \alpha = 0.05,\ B = 10.0,\ x_c = 1.0$$

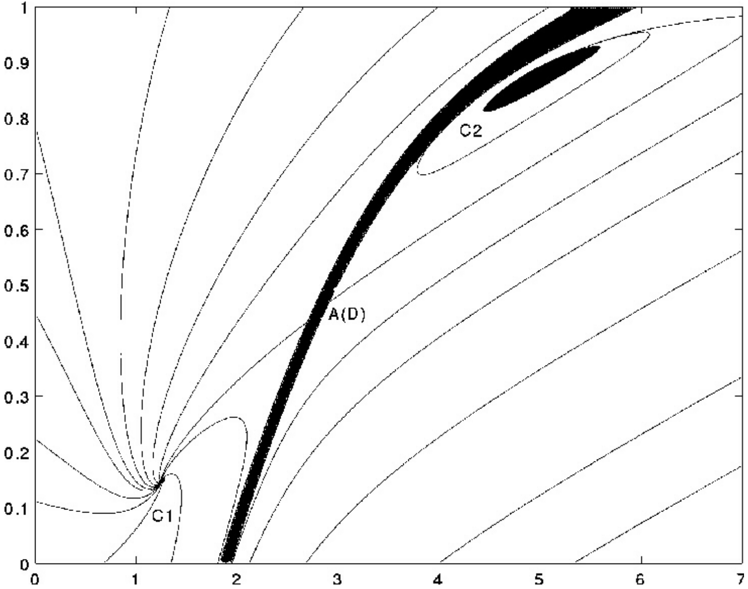
and control range

$$\Omega = [-0.15, 0.15].$$

The uncontrolled system has an unstable (hyperbolic) fixed point at

$$(x_1^*, x_2^*) \sim (2.8, 0.45) \in D$$


C2
D
C1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7



1
0.9
0.8
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0.6
0.5
0.4
0.3
0.2
0.1
0
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2
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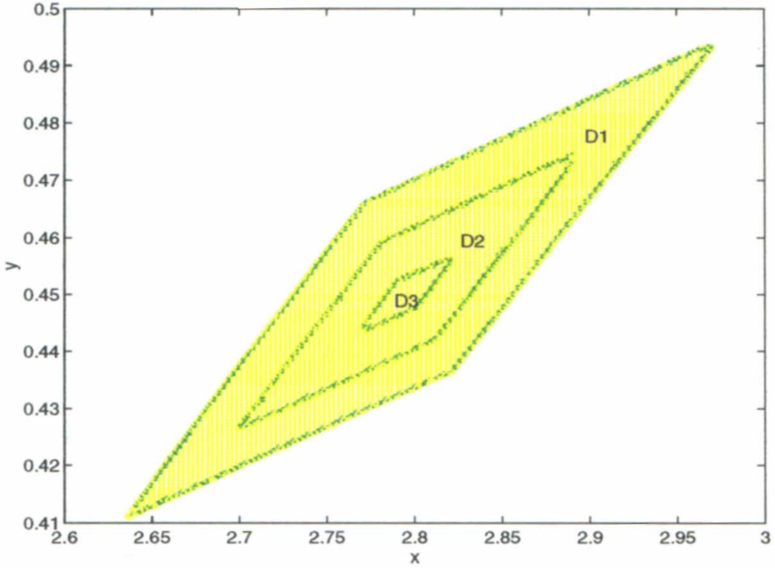
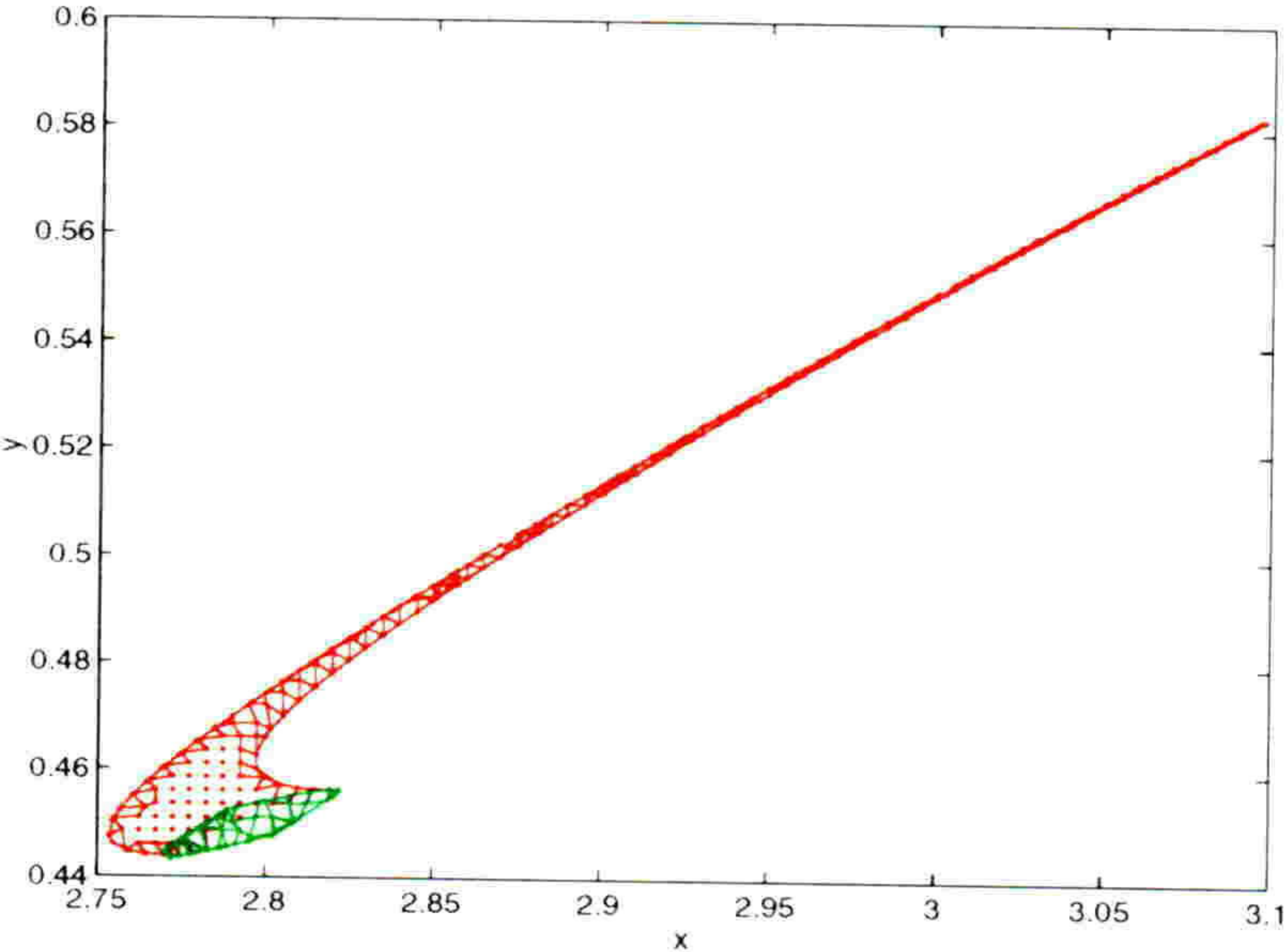
4
5
6
7
C2
A(D)
C1

Stabilization by piecewise constant feedbacks

Let D be the control set around the equilibrium with control range $\Omega = [-\rho, \rho]$. Define

$$
\begin{aligned}
R_0 \quad &: \quad = \{x \,|\, \varphi(t,x,\rho) \in D \text{ for some } t > 0\}, \\
R_1 \quad &: \quad = \{x \,|\, \varphi(t,x,-\rho) \in R_0 \cup D \text{ for some } t > 0\} \setminus R_0 \\
R_2 \quad &: \quad = \{x \,|\, \varphi(t,x,\rho) \in R_0 \cup R_1 \cup D \text{ for some } t > 0\} \setminus (R_0 \cup R_1) \\
& \quad \text{etc.}
\end{aligned}
$$


0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
y
2.6
2.65
2.7
2.75
2.8
2.85
2.9
2.95
3
x
D1
D2
D3


0.6
0.58
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y
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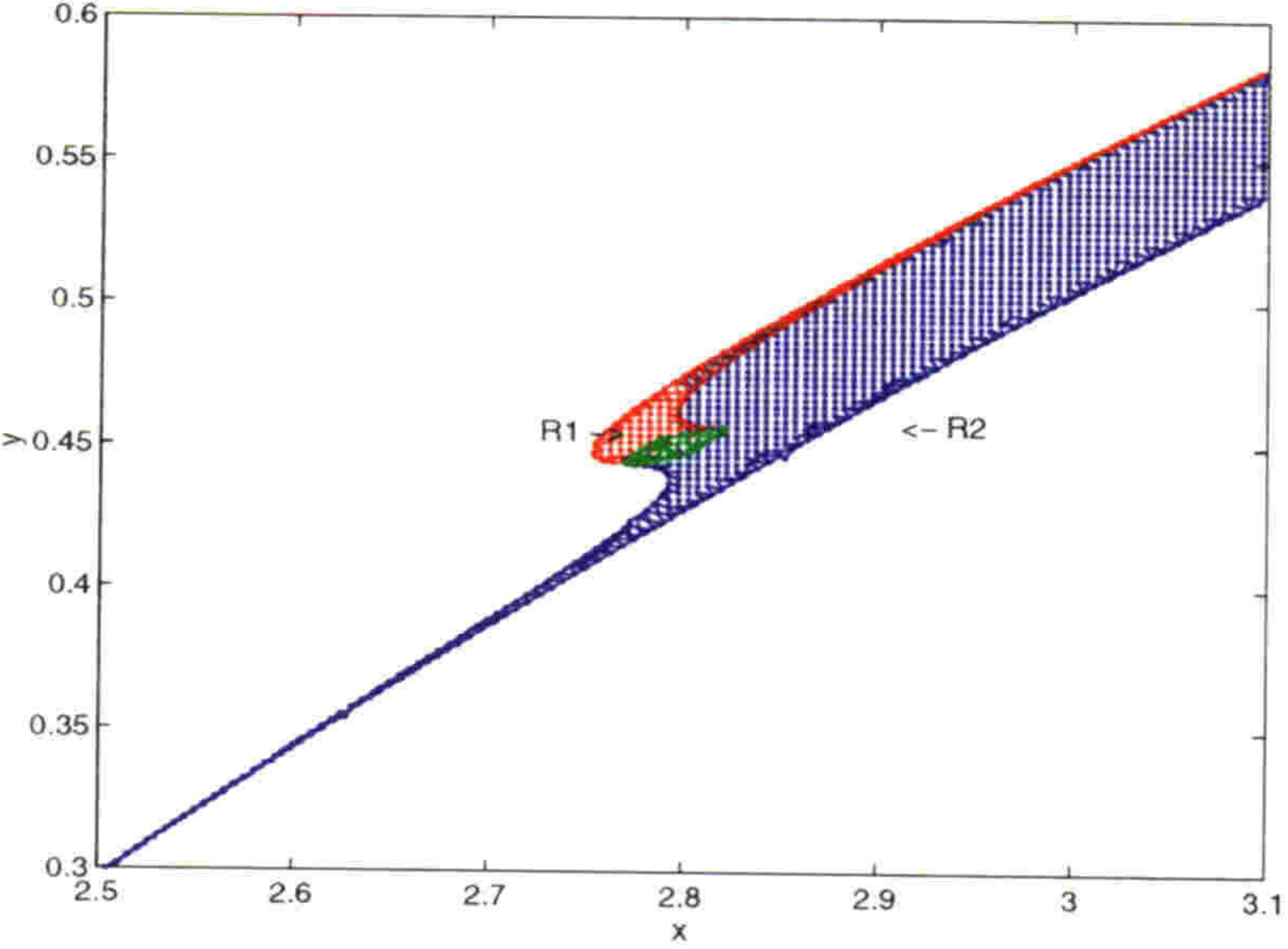
2.95
3
3.05
3.1
x


0.6
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y 0.45
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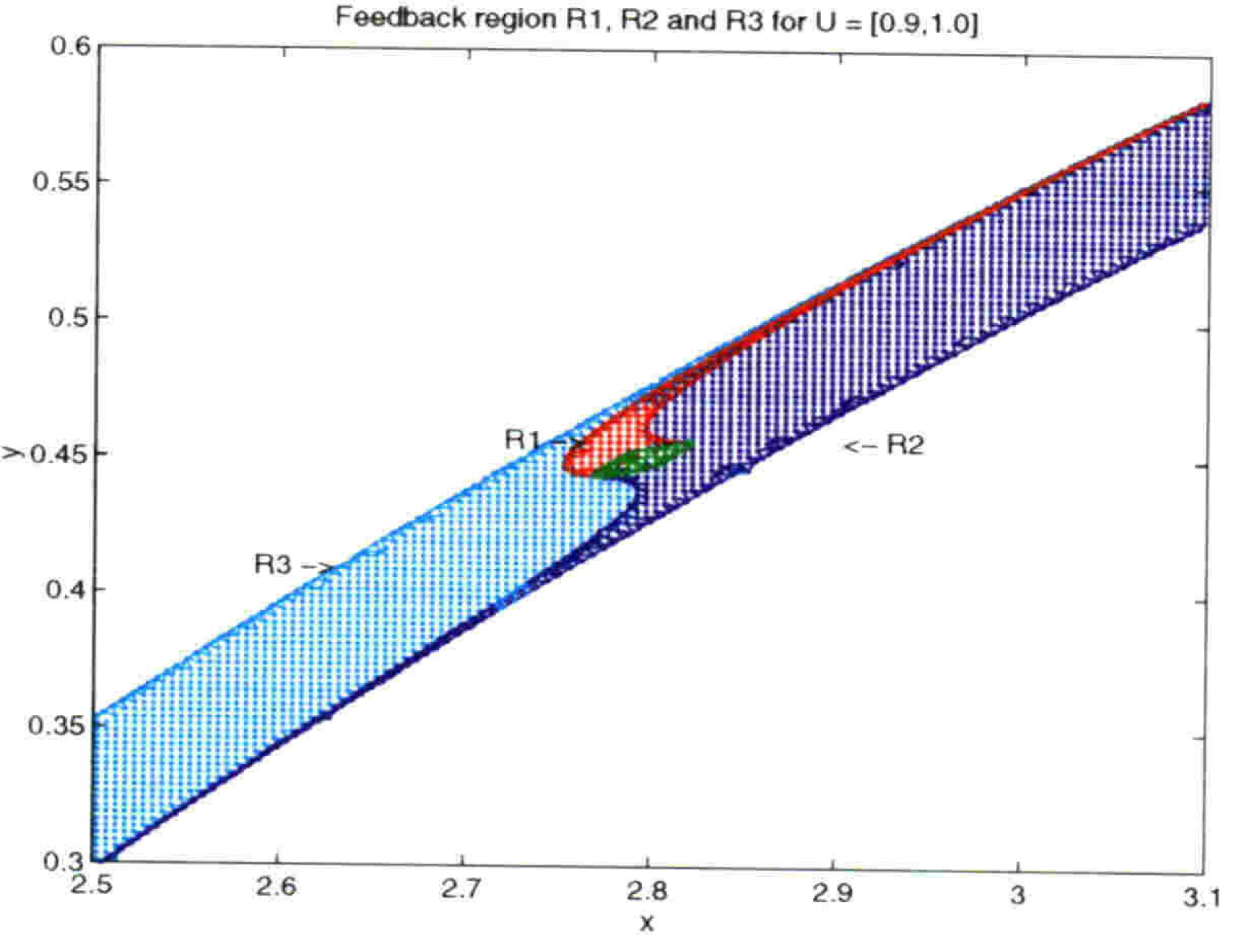
2.7
2.8
2.9
3
3.1
x
R1 ->
<- R2


Feedback region R1, R2 and R3 for U = [0.9,1.0]
0.6
0.55
0.5
0.45
0.4
0.35
0.3
y
2.5
2.6
2.7
2.8
2.9
3
3.1
x
R1 ->
<- R2
R3 ->

Final remarks

Since asymptotic stabilization is a basic problem in control, there is a multitude of algorithms to achieve it, in addition to the concepts presented above.

- Backstepping
- Model-predictive control (receding horizon optimal control)
- ...

Note also, that in applications stability is only one goal among others including, in particular, robustness properties with respect to perturbations and overshoot.