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# Spectral Theory for Bilinear Control Systems

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# Introduction

A bilinear control systems has the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t) = A(u)x, \; u(t) = (u_i(t))_{i=1,\ldots,m} \in \Omega,$$

with $d \times d$-matrices $A_0, A_1, \ldots, A_m \in \mathbb{R}^{d \times d}$ and compact convex control range $\Omega \subset \mathbb{R}^m$.

We will consider the associated control flow and controllability properties as well as exponential stability properties.

Crucial insight will be gained by analyzing the projection to (real) projective space $\mathbb{P}^{d-1}$.

Different approaches to bilinear control systems can be found e.g. in
D.L. Elliott, Bilinear Control Systems, 2009
San Martin/Seco, Erg.Th.Dyn.Syst.(2010) based on semigroups in Lie groups.

# The linear oscillator

The linear oscillator with control/uncertainty in the restoring force:

$$\ddot{x} + b\dot{x} + [1 + u(t)]x = 0, \text{ with } u(t) \in [-\rho, \rho], b = 1.5 > 0.$$

or, in state space form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$.

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- Linear control flows

- Lyapunov exponents and the projected system

- Selgrade's Theorem and chain control sets

- Proof of Selgrade's Theorem: Morse decompositions and attractor-repeller pairs

- the Morse spectrum

- control sets versus chain control sets

# The linear control flow

As in the general case, a bilinear control system defines a control flow on $\mathcal{U} \times \mathbb{R}^d$, given by

$$\Phi(t, u, x) = (\theta_t u, \varphi(t, x, u)), t \in \mathbb{R}.$$

The special property of this control flow is its linearity with respect to $x$,

$$\Phi(t, u, \alpha x + \beta y) = \alpha\Phi(t, u, x) + \beta\Phi(t, u, y), \alpha, \beta \in \mathbb{R}.$$

The state space $\mathcal{U} \times \mathbb{R}^d$ has the structure of a (topologically trivial) vector bundle with compact metric base space $\mathcal{U}$.
Furthermore, we know that the periodic points are dense for the shift $\theta$, hence the base space is chain transitive.

# Projective space

Linearity of $\Phi(t, u, x)$ in $x$ immediately implies that one gets an induced flow on $\mathcal{U} \times \mathbb{P}^{d-1}$.
$\mathbb{P}^{d-1}$ may be obtained by identifying opposite points on the unit sphere.
For a solution $x(t) = \varphi(t, x_0, u)$ of $\dot{x} = A(u)x$ one obtains with $s(t) = \frac{x(t)}{\|x(t)\|}$, where $\|x(t)\| = \sqrt{\langle x(t), x(t) \rangle}$,

$$\dot{s}(t) = \left[ A(u) - s(t)^T A(u) s(t) \cdot I \right] s(t).$$

In fact,

$$\begin{aligned} \dot{s} &= \frac{\dot{x}\, \|x\| - x \langle \dot{x}, x \rangle / \|x\|}{\|x\|^2} = \frac{A(u)x\, \|x\| - x \langle A(u)x, x \rangle / \|x\|}{\|x\|^2} \\ &= \left[ A(u) - s(t)^T A(u) s(t) \cdot I \right] s(t). \end{aligned}$$

Abbreviating $h(s, u) = \left[ A(u) - s^T A(u) s \cdot I \right] s$ we can write this as

$$\dot{s}(t) = h(s(t), u(t)) \text{ on } \mathbb{S}^{d-1}.$$

The subtracted term $\left[ s^T A(u) s \right] s$ is the radial component of $A(u)s$.

# Exponential growth rates

The exponential growth rate or **Lyapunov exponent** of a solution for $(u, x_0)$ is

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \left\| \varphi(t, x_0, u) \right\| .$$

Somewhat surprisingly, also the Lyapunov exponents are determined by the induced system on projective space,

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau)) d\tau \text{ with } q(u, s) := s^\top A(u) s.$$

# Selgrade's Theorem

**Theorem.** Let $\Phi$ be a continuous linear flow on on a vector bundle $\mathcal{U} \times \mathbb{R}^d$ with compact chain transitive base space $\mathbb{R}^d$. Then the induced flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ has only finitely many chain recurrent components $\mathcal{M}_1, \ldots, \mathcal{M}_\ell, 1 \leq \ell \leq d$. They have the following form:

Every $\mathcal{M}_i$ defines an invariant subbundle via

$$\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i) = \{(u, x) \in \mathcal{U} \times \mathbb{R}^d \,|(u, \mathbb{P}x) \in \mathcal{M}_i\}$$

and the following decomposition into a Whitney sum holds

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\ell.$$

# A simple example

Consider the linear autonomous ordinary differential equation

$$\dot{x} = Ax.$$

For an eigenvector $x$ corresponding to a real eigenvalue $\mu$ of $A$ the point $\mathbb{P}x$ is an equilibrium in $\mathbb{P}^{d-1}$.

More generally, let $\lambda_1, \ldots, \lambda_\ell$ be the pairwise different real parts of the eigenvalues of $A$ and denote by $V(\lambda_i)$ be the direct sum of all generalized eigenspaces for the eigenvalues with real part equal to $\lambda_i$. Then the $\mathcal{M}_i := \mathbb{P}V_i$ are the chain recurrent components and

$$\mathbb{R}^d = \bigoplus_{i=1}^{\ell} V(\lambda_i) = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{M}_i.$$

# The chain control sets

**Corollary**. For a bilinear control system $\dot{x} = A(u)x$, $u(t) \in \Omega$, there are $1 \leq \ell \leq d$ chain control sets $E_i$ for the induced system in projective space and there is a Whitney decomposition

$$\mathcal{U} \times \mathbb{R}^d = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{E}_i,$$

where the $\mathcal{E}_i$ are the lifts of the chain control sets $E_i$ in $\mathbb{P}^{d-1}$,

$$\mathcal{E}_i = \left\{(u,p) \in \mathcal{U} \times \mathbb{P}^{d-1} \,\middle|\, s(t,p,u) \in E_i \text{ for } t \in \mathbb{R}\right\},$$

with $s(t,p,u)$ denoting the solution of

$$\dot{s}(t) = \left[A - s(t)^T A s(t) \cdot I\right] s(t), s(0) = p.$$

# Questions:

- Proof of Selgrade's theorem

- How are the Lyapunov exponents related to the chain control sets?

- Do the chain control sets coincide with the control sets in projective space?

- What about the control sets in $\mathbb{R}^d$ ?

- Consequences for stability and stabilizability

# On the proof of Selgrade's theorem

This is based on the relation between chain recurrence, Morse decompositions and attractor-repeller pairs.

**Recall:**
A **Morse decomposition** of a flow is given by $\{\mathcal{M}_i \,|\, i = 1, \ldots, \ell\}$ with nonvoid, pairwise disjoint and compact isolated invariant sets s.t.
(i) $\forall x \in X : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$;
(ii) there are no cycles.

If the number of chain recurrent components is finite, this corresponds to the finest Morse decomposition. In particular, if the number of chain control sets in a compact invariant set is finite, this corresponds to the finest Morse decomposition of the control flow.

**Definition.** For a flow on a compact metric space $X$ an **attractor** $A$ is a compact invariant set with a nbhd $N$ such that

$$A = \omega(N) := \{y \in X \,|\, \exists (x_n) \in N, \exists t_n \to \infty : y = \lim x_n \cdot t_n\}.$$

A compact invariant set $R$ is a **repeller** if it has a nbhd $N^*$ such that

$$R = \alpha(N^*) := \{y \in X \,|\, \exists (x_n) \in N^*, \exists t_n \to -\infty : y = \lim x_n \cdot t_n)\}.$$

**Proposition.** For every attractor $A$

$$A^* := \{x \in X \,|\, \omega(x) \cap A = \varnothing\}$$

is a repeller, called the complementary repeller.

# Morse decompositions and attractor-repeller pairs

**Theorem.** Let $\mathcal{M}_i$, $i = 1, \ldots, n$, be subsets of $X$. Equivalent are:

(i) $\{\mathcal{M}_i \,|i = 1, \ldots, \}$ form a Morse decomposition;

(ii) there is an increasing sequence of attractors

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ for $0 \leq i \leq n-1$.

**Example.**

$$\dot{x} = x(x-1)(x-2)^2(x-3).$$

A Morse decomposition is given by

$$\mathcal{M}_1 = \{0\} \preceq \mathcal{M}_3 = \{1\} \succeq \mathcal{M}_2 = [2,3].$$

Here $n = 3, A_0 = \varnothing, A_0^* = [0,3], A_1 = \{1\}, A_1^* = \{0\} \cup [2,3]$,
$A_2 = [1,3], A_2^* = \{0\}, A_3 = [0,3], A_3^* = \varnothing$ and

$$A_1 \cap A_0^* = \{1\} = \mathcal{M}_3, A_2 \cap A_1^* = [2,3] = \mathcal{M}_2, A_3 \cap A_2^* = \{0\} = \mathcal{M}_1.$$

# Proof of Selgrade's theorem

Steps of the proof: Show that

- an attractor for the projectivized flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ defines a (linear!) subbundle of $\mathcal{U} \times \mathbb{P}^d$.

- an attractor-repeller pair defines an invariant subbundle decomposition for the linear flow $\Phi$ on $\mathcal{U} \times \mathbb{R}^d$.

- then one can use the dimension of the subbundles to show that there is a finest Morse decomposition into Morse sets $\mathcal{M}_i$, hence

- this are the chain recurrent components in $\mathcal{U} \times \mathbb{P}^d$

- defining a decomposition of $\mathcal{U} \times \mathbb{R}^d$ into invariant subbundles $\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i)$.

# The Morse spectrum of the bilinear system I

Recall: For $\varepsilon, T > 0$ an $(\varepsilon, T)$-chain $\zeta$ in $\mathcal{U} \times \mathbb{P}^{d-1}$ is given by

$$n \in \mathbb{N}, T_0, T_1, \ldots, T_{n-1} > T, (u_0, p_0), \ldots, (u_n, p_n) \in \mathcal{U} \times \mathbb{P}^{d-1}$$

such that

$$d(\mathbb{P}\Phi(T_i, (u_i, p_i)), (u_{i+1}, p_{i+1})) < \varepsilon \text{ for all } i.$$

With $\mathbb{P}x_i = p_i$ define the chain exponent of $\zeta$ as

$$\lambda(\zeta) = \left(\sum_{i=1}^{n-1} T_i\right)^{-1} \sum_{i=1}^{n-1} \left(\log \|\varphi(T_i, x_i, u_i)\| - \log \|x_i\|\right),$$

The **Morse spectrum** is

$$\Sigma_{Mo} = \{\lambda \in \mathbb{R} \,|\, \exists \varepsilon_n \to 0, \exists T_n \to \infty, (\varepsilon_n, T_n)\text{-chains } \zeta_n : \lim \lambda(\zeta_n) = \lambda\}.$$

# The Morse spectrum of the bilinear system II

**Theorem:**

(i) $\Sigma_{Mo} = \bigcup_{i=1}^{\ell} \Sigma_{Mo}(\mathcal{M}_i)$

(ii) Each $\Sigma_{Mo}(\mathcal{M}_i)$ consists of a closed interval $[\kappa_i^*, \kappa_i]$.

(iii) For $i < j$ we have $\kappa_i^* < \kappa_j^*$ and $\kappa_i < \kappa_j$.

(iv) $\Sigma_{Ly} \subset \Sigma_{Mo}$ and the $\kappa_i^*, \kappa_i$ are actually Lyapunov exponents.

# (Un)stable subbundle

The upper spectral interval $\Sigma_{Mo}(\mathcal{M}_\ell) = [\kappa_\ell^*, \kappa_\ell]$ determines the robust stability of $\dot{x} = A(u(t))x$ (and stabilizability of the system if the set $\mathcal{U}$ is interpreted as a set of admissible control functions).

The **stable, center, and unstable subbundles** of $\mathcal{U} \times \mathbb{R}^d$ are defined as

$$L^- = \bigoplus_{j:\ \kappa_j < 0} \mathcal{V}_j,\ \ L^0 = \bigoplus_{j:\ 0 \in [\kappa_j^*, \kappa_j]} \mathcal{V}_j, L^+ = \bigoplus_{j:\ \kappa_j^* > 0} \mathcal{V}_j.$$

**Corollary.** The zero solution of $\dot{x} = A(u(t))x$, $u \in \mathcal{U}$, is exponentially stable for all $u \in \mathcal{U}$ iff $\kappa_\ell < 0$ iff $L^- = \mathcal{U} \times \mathbb{R}^d$.

# Stability radius

Suppose that $0 \in \mathrm{int}\Omega$ and consider the control ranges $\Omega^\rho := \rho\Omega$.

The maximal spectral value $\kappa_\ell(\rho)$ is continuous in $\rho$ and we define the (asymptotic-) stability radius of this family as

$$\begin{aligned} r &= \inf\{\rho \geq 0 \,|\, \exists u \in \mathcal{U}^\rho : \dot{x}^\rho = A(u(t))x^\rho \text{ is not exp. stable}\} \\ &= \inf\{\rho \geq 0 \,|\, \kappa_\ell(\rho) > 0\} \end{aligned}$$

# The linear oscillator

The linear oscillator with control/uncertainty in the restoring force:

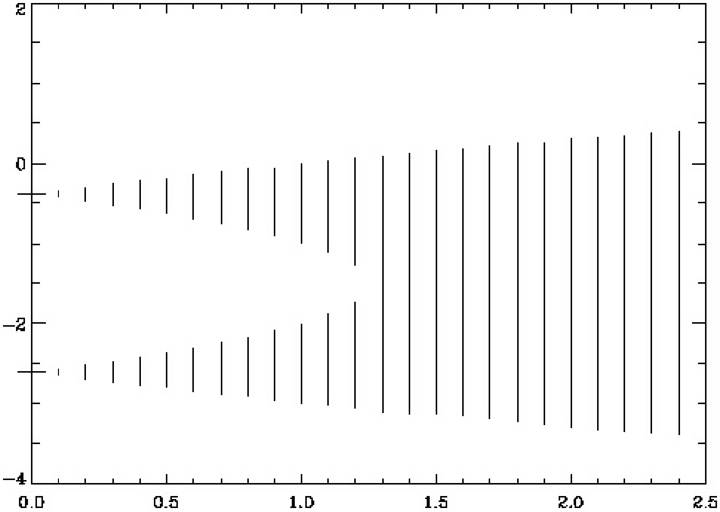
$$\ddot{x} + b\dot{x} + [1 + u(t)]x = 0, \text{ with } u(t) \in [-\rho, \rho], b = 1.5 > 0.$$

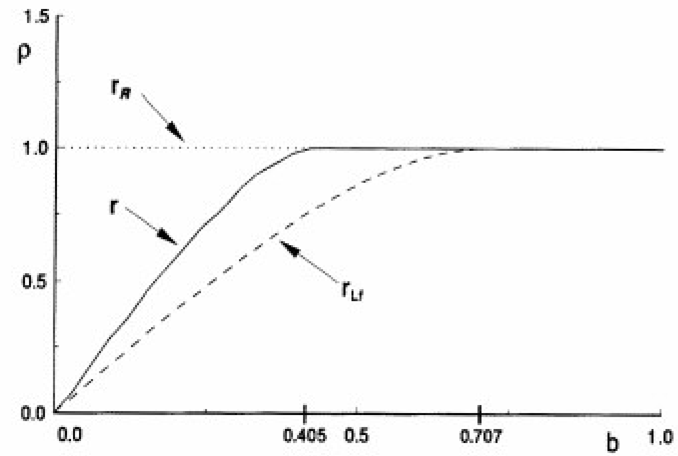
or, in state space form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$. (For $b \leq 0$ the system is unstable even for constant perturbations.)

# Spectral intervals for the linear oscillator





1.5
ρ
r_R
1.0
r
0.5
r_Lf
0.0
0.0
0.405
0.5
0.707
b
1.0

# Control sets and chain control sets

**Theorem.** Assume that the Lie algebra rank condition for the system on $\mathbb{P}^{d-1}$ holds.

(i) Every chain control set contains a control set $D_j$ with nonvoid interior.

(ii) There are $k$ control sets $D_j$ with nonvoid interior, $0 < \ell \leq k \leq d$, in $\mathbb{P}^{d-1}$.

(ii) Exactly one of them is an invariant control set.

# An example: control sets vs chain control sets

Consider the bilinear control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + u_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u_2(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\Omega = [0, \frac{1}{2}] \times [1, 2]$. For $u_1 \equiv 0,\ u_2 \equiv 1$ we have a double eigenvalue $\lambda_{1,2} = 1$ of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with eigenspace $\mathbb{R}^2$, hence there is a single chain control set $E = \mathbb{P}^1$.
There are two control sets in $\mathbb{P}^1$ given by

$$D_1 = \left(-\frac{\pi}{4}, 0\right) \text{ and } D_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

# Control sets and chain control sets

Suppose that for $\rho < \rho'$, i.e. for increasing control ranges $\rho\Omega \subset \rho'\Omega$, the reachable sets in $\mathbb{P}^{d-1}$ are strictly increasing.

Then for all up to at most $d-1$ $\rho$-values the closures of the control sets are the chain control sets and the spectral growth rates satisfy

$$\Sigma_{Ly}(\mathbb{P}D_j) = \Sigma_{Mo}(E_j).$$

The control sets of the bilinear system in $\mathbb{R}^d$ are exactly those cones over the control sets $\mathbb{P}D_j$ for which $0 \in (\kappa_j^*, \kappa_j)$.

The bilinear system is completely controllable in $\mathbb{R}^d \setminus \{0\}$ iff the projected system is completely controllable and $0 \in (\kappa^*, \kappa)$.

# Concluding remarks

Bilinear control systems may be viewed as linear flows on vector bundles.

Their topological analysis via chain transitivity, Morse decompositions and attractors leads to a spectral theory which allows us to find results on controllability and stability.
There are further consequences on stabilizability by (time varying) feedbacks.