University of Teheran
January 2019

Control under Communication Constraints and Invariance Entropy

Fritz Colonius
Universität Augsburg

Goal

Determine **fundamental limitations** in control

Here: Describe the "information" needed to make a subset invariant for a control system

Classically, entropy is used in dynamical systems theory in order to describe the information generated by the systems and to classify them.

A recent survey on various definitions and application areas of **entropy** is Amigó et al. DCDS B (2015).

Control systems:

> **Delchamps** (1990) (ergodic theory for quantized feedback)

Topological versions have been analyzed, in particular, by

> **Nair, Evans, Mareels and Moran** (2004)
>
> **Kawan**, Springer LNM Vol. 2089 (2013)

Control systems

We consider control system in discrete time given by

$$x_{n+1} = f(x_n, u_n), n \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous and M and Ω are metric spaces. The solution with $x_0 = x$ and $u = (u_n) \in \mathcal{U} := \Omega^N$ is denoted by $\varphi(n, x, u)$, $n \in \mathbb{N}$.

We assume that for every $x \in Q \subset M$ there is $u(x) \in \Omega$ with $f(x, u(x)) \in Q$.
What is the "**information**" necessary to keep the system in Q?

Control systems

We consider control system in discrete time given by

$$x_{n+1} = f(x_n, u_n), n \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous and M and Ω are metric spaces. The solution with $x_0 = x$ and $u = (u_n) \in \mathcal{U} := \Omega^N$ is denoted by $\varphi(n, x, u)$, $n \in \mathbb{N}$.

We assume that for every $x \in Q \subset M$ there is $u(x) \in \Omega$ with $f(x, u(x)) \in Q$.
What is the "**information**" necessary to keep the system in Q?

Motivation: Suppose that the present state x_n of the system is measured. If the controller has complete information about the present state, it can adjust a feedback control $u(x)$ appropriately. However, if the measurement is sent to the controller via a (noiseless) digital channel with bounded data rate it is of interest to determine the minimal data rate needed to make Q invariant. More abstractly: What is the minimal average information needed to make Q invariant?

Contents

This talk consists of three parts:

- Some motivation from classical entropy of dynamical systems

- Topological invariance entropy for control systems

- coder-controllers and minimal bit rates

- Relations to controllability properties

Topological entropy for dynamical systems

Let $T : X \to X$ be a continuous map on a compact metric space.
Suppose $\mathcal{B}$ is a finite open cover of X, i.e., the sets in $\mathcal{B}$ are open, their union is X.
For an **itinerary** $\alpha = (B_0, B_1, \ldots, B_{n-1}) \in \mathcal{B}^n$ let

$$\mathcal{B}_n(\alpha) = \{x \in X \,|\, T^j(x) \in B_j \text{ for } j = 0, \ldots, n-1\} = B_0 \cap \cdots \cap T^{-(n-1)} B_{n-$$

They again form an open cover of X,

$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$.

Topological entropy for dynamical systems

Let $T : X \to X$ be a continuous map on a compact metric space.
Suppose $\mathcal{B}$ is a finite open cover of X, i.e., the sets in $\mathcal{B}$ are open, their union is X.
For an **itinerary** $\alpha = (B_0, B_1, \ldots, B_{n-1}) \in \mathcal{B}^n$ let

$$\mathcal{B}_n(\alpha) = \{x \in X \,|\, T^j(x) \in B_j \text{ for } j = 0, \ldots, n-1\} = B_0 \cap \cdots \cap T^{-(n-1)} B_{n-}$$

They again form an open cover of X,

$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$.
Then the entropy of $\mathcal{B}$ is given by

$$h(\mathcal{B}, T) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)})$$

and the **topological entropy** of T is

$$h_{top}(T) = \sup_{\mathcal{B}} h(\mathcal{B}, T).$$

Adler, Konheim, McAndrew (1965)

A classical example

Consider the **logistic map** on the interval $X = [0, 1]$ given by

$$F_4(x) = 4x(1 - x), x \in [0, 1].$$

The topological entropy of F_4 is

$$h_{top}(F_4) = \log_2 2 = 1 > 0.$$

Hence this is a **chaotic map**.

Metric entropy for dynamical systems

For a probability measure μ and a partition $\mathcal{P}$ of X the **Shannon entropy** is

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

Let μ be invariant for a map T on X, i.e., $\mu(T^{-1}B) = \mu(B)$ for all $B \subset X$.

Metric entropy for dynamical systems

For a probability measure μ and a partition $\mathcal{P}$ of X the **Shannon entropy** is

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

Let μ be invariant for a map T on X, i.e., $\mu(T^{-1}B) = \mu(B)$ for all $B \subset X$.

For an **itinerary** $\alpha = (P_0, P_1, ..., P_{n-1}) \in \mathcal{P}^n$ let

$$P_n(\alpha) = \{x \in X \,|\, T^j(x) \in P_j \text{ for all } j\} = P_0 \cap T^{-1}P_1 \cap \cdots \cap T^{-(n-1)}P_{n-1}.$$

They yield a partition $\mathcal{P}^{(n)} = \{P_n(\alpha)\,|\alpha \in \mathcal{P}^n\}$ and

$$h_\mu(\mathcal{P}, T) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\mathcal{P}^{(n)}\right).$$

The **Kolmogorov-Sinai entropy** of T is

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}, T).$$

The logistic map again

Recall

$$F_4(x) = 4x(1-x) \text{ on } [0,1].$$

A (trivial) invariant measure is $\mu = \delta_0$ with entropy $h_{\delta_0}(F_4) = 0$.
A nontrivial invariant measure is given by its density (with respect to Lebesgue measure)

$$\frac{1}{\pi\sqrt{x(1-x)}}, x \in [0,1].$$

The corresponding **metric entropy** is

$$h_\mu(F_4) = \log_2 2 = 1$$

(hence equal to the topological entropy).

Comments

The **Variational Principle** states that

$$\sup_{\mu} h_{\mu}(T) = h_{top}(T)$$

and invariant measures μ with maximal entropy, i.e., $h_{\mu}(T) = h_{top}(T)$, are of special relevance.

For smoth maps, the entropy can often be characterized by the positive **Lyapunov exponents**.

Invariance entropy for control systems

Describe the **minimal information** to make a compact $Q \subset M$ invariant for

$$x_{n+1} = f(x_n, u_n),\ \ u_n \in \Omega,$$

with solutions $\varphi(n, x_0, u)$, $n \in \mathbb{N}$, in M.

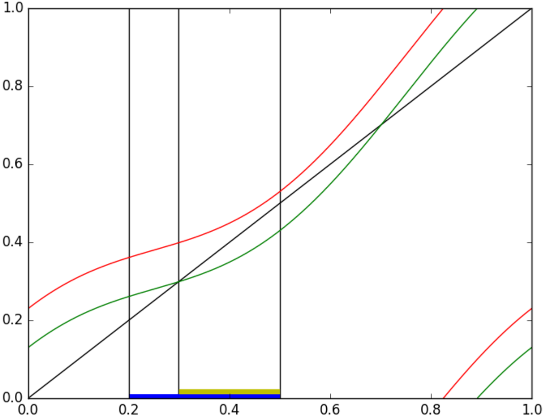
Here this will be done in a topological framework.
Topological invariance entropy is based on **itineraries in Q** corresponding to **invariant open covers** of Q. They are constructed by feedbacks keeping the system in Q and replace the open covers.

Observe: This is not directly related to the entropy of the uncontrolled system which may behave very wildly in Q, while Q itself may be invariant. Hence the entropy of the dynamical system may be positive while the invariance problem is trivial.

Example

$$f_\alpha(x,\omega) = x + \sigma\cos(2\pi x) + A\omega + \alpha \mod 1,\ \omega \in \Omega = [-1,1].$$

With $A = 0.05, \sigma = 0.1, \alpha = 0.08$ consider the set $Q = [0.2, 0.5]$.



An **invariant open cover** $\mathcal{C} = (, \mathcal{B}, F)$ is given by $\tau \in \mathbb{N}$, an open cover $\mathcal{B}$ of Q and $F : \mathcal{B} \to \Omega^\tau$ with

$$\varphi(j, B, F(B)) \subset \mathrm{int} Q \text{ for } j = 1, \ldots, \tau \text{ and } B \in \mathcal{B}.$$

Topological invariance entropy for control systems

An **invariant open cover** $\mathcal{C} = (\tau, \mathcal{B}, F)$ is given by $\tau \in \mathbb{N}$, an open cover $\mathcal{B}$ of Q and $F : \mathcal{B} \to \Omega^\tau$ with

$$\varphi(j, B, F(B)) \subset \mathrm{int}Q \text{ for } j = 1, \ldots, \tau \text{ and } B \in \mathcal{B}.$$

For a **$\mathcal{C}$-itinerary** $\alpha = (B_0, \ldots, B_{n-1}) \in \mathcal{B}^n$ define $u_\alpha = (F(B_0), F(B_1), \ldots)$ and

$$B_n(\alpha) = \{x \in Q \,|\, \varphi(i\tau, x, u_\alpha) \in B_i \text{ for } i = 0, \ldots, n-1\}.$$

These sets again form an open cover of Q,

$$\mathfrak{B}^{(n)} = \{B_n(\alpha) \,|\, \alpha \in \mathcal{B}^n\}.$$

Topological invariance entropy for control systems

An **invariant open cover** $\mathcal{C} = (\tau, \mathcal{B}, F)$ is given by $\tau \in \mathbb{N}$, an open cover $\mathcal{B}$ of Q and $F : \mathcal{B} \to \Omega^\tau$ with

$$\varphi(j, B, F(B)) \subset \operatorname{int} Q \text{ for } j = 1, \ldots, \tau \text{ and } B \in \mathcal{B}.$$

For a **$\mathcal{C}$-itinerary** $\alpha = (B_0, \ldots, B_{n-1}) \in \mathcal{B}^n$ define
$u_\alpha = (F(B_0), F(B_1), ...)$ and

$$B_n(\alpha) = \{x \in Q \,|\, \varphi(i\tau, x, u_\alpha) \in B_i \text{ for } i = 0, \ldots, n-1\}.$$

These sets again form an open cover of Q,

$$\mathfrak{B}^{(n)} = \{B_n(\alpha) \,|\, \alpha \in \mathcal{B}^n\}.$$

The invariance entropy of $\mathcal{C}$ is

$$h(\mathcal{C}, Q) := \lim_{n \to \infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)})$$

and the **topological invariance entropy** of Q is

$$h_{inv}(Q) := \inf_{\mathcal{C}} h(\mathcal{C}, Q).$$

Alternative definition

Let K be a subset of $Q \subset M$ suh that for all $x \in K$ there is $u \in \mathcal{U}$ with $\varphi(n, x, u) \in Q$ for all $n \in \mathbb{N}$.
A subset $\mathcal{S} \subset \mathcal{U}$ is called (τ, K, Q)-spanning, if for all $x \in K$ there is $u \in \mathcal{S}$ such that for all $j = 1, \ldots, \tau$

$$\varphi(t, x, u) \in \mathrm{int}Q \text{ (or } \varphi(t, x, u) \in Q).$$

Thus $\mathcal{U}$ is (τ, K, Q)-spanning for all $\tau \in \mathbb{N}$.
Let $r_{inv}(\tau, K, Q)$ be the minimal number of elements in a (τ, K, Q)-spanning set.
The invariance entropy of (K, Q) is

$$h_{inv}(K, Q) := \lim_{\tau \to \infty} \log r_{inv}(\tau, K, Q).$$

Alternative definition

Let K be a subset of $Q \subset M$ suh that for all $x \in K$ there is $u \in \mathcal{U}$ with $\varphi(n, x, u) \in Q$ for all $n \in \mathbb{N}$.
A subset $\mathcal{S} \subset \mathcal{U}$ is called (τ, K, Q)-spanning, if for all $x \in K$ there is $u \in \mathcal{S}$ such that for all $j = 1, \ldots, \tau$

$$\varphi(t, x, u) \in \mathrm{int} Q \text{ (or } \varphi(t, x, u) \in Q).$$

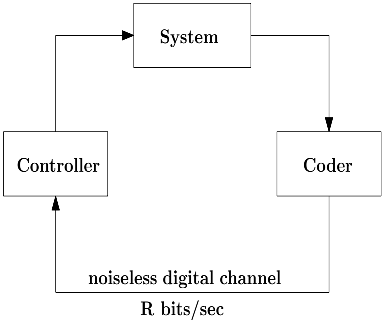
Thus $\mathcal{U}$ is (τ, K, Q)-spanning for all $\tau \in \mathbb{N}$.
Let $r_{inv}(\tau, K, Q)$ be the minimal number of elements in a (τ, K, Q)-spanning set.
The invariance entropy of (K, Q) is

$$h_{inv}(K, Q) := \lim_{\tau \to \infty} \log r_{inv}(\tau, K, Q).$$

Theorem (FC, Kawan, Nair (2013)). For $K = Q$ one has

$$h_{inv}(Q) = h_{inv}(Q, Q).$$

The control problem



Explanation

System Deterministic, discrete or continuous time
Coder Encodes the state by a symbol from a (time-dependent) alphabet at discrete times $k\tau$, $k = 0, 1, 2, \ldots$
Controller Generates open-loop controls on a finite time interval $[0, \tau]$

Relation to coder-controllers and data rates

A **coder-controller** has the form $\mathcal{H} = (S, \gamma, \delta, \tau)$ where
- $S = (S_k)_{k \in \mathbb{N}}$ denotes finite coding alphabets
- the coder mapping $\gamma_k : M^{k+1} \to S_k$ associates to the present and past states the symbol $s_k \in S_k$
- at time $k\tau$ the controller mapping is $\delta_k : S_0 \times \cdots \times S_k \to \Omega^\tau$.

The **transmission data rate** is

$$R(\mathcal{H}) = \liminf_{k \to \infty} \frac{1}{k\tau} \sum_{j=0}^{k-1} \log \# S_j.$$

A **coder-controller** has the form $\mathcal{H} = (S, \gamma, \delta, \tau)$ where
- $S = (S_k)_{k \in \mathbb{N}}$ denotes finite coding alphabets
- the coder mapping $\gamma_k : M^{k+1} \to S_k$ associates to the present and past states the symbol $s_k \in S_k$
- at time $k\tau$ the controller mapping is $\delta_k : S_0 \times \cdots \times S_k \to \Omega^\tau$.

The **transmission data rate** is

$$R(\mathcal{H}) = \liminf_{k \to \infty} \frac{1}{k\tau} \sum_{j=0}^{k-1} \log \#S_j .$$

$\mathcal{H}$ renders Q invariant if for every $x_0 \in Q$ the sequence

$$x_{k+1} := \varphi(\tau, x_k, u_k), k \in \mathbb{N},$$

with

$$u_k = \delta_k(\gamma_0(x_0), \gamma_1(x_0, x_1), \ldots, \gamma_k(x_0, x_1, \ldots, x_k)) \in \Omega^\tau$$

satisfies

$$\varphi(i, x_k, u_k) \in Q \text{ for all } i \in \{1, \ldots, \tau\} \text{ and all } k \in \mathbb{N}.$$

The data rate theorem

Theorem. For a compact and controlled invariant set Q it holds that

$$h_{inv}(Q) = \inf R(\mathcal{H}),$$

where the infimum is taken over all coder-controllers $\mathcal{H}$ that render Q invariant.

Comments and some further results

- Let $K_1, K_2 \subset D$ be compact with $\mathrm{int} K_i \neq \varnothing$ in a control set D. Then $h_{inv}(K_1, Q) = h_{inv}(K_2, Q)$.

- For **linear control systems** in $\mathbb{R}^d$

$$x_{n+1} = Ax_n + Bu_n, u_n \in \Omega \subset \mathbb{R}^m,$$

with $\mathrm{int} K \neq \varnothing$ and (A, B) controllable, A hyperbolic and Ω a compact nbhd of 0, one has for K contained in the unique control set D

$$h_{inv}(K, D) = \sum_{\lambda \in \sigma(A)} \max(0, \log |\lambda|).$$

Comments and some further results

- Let $K_1, K_2 \subset D$ be compact with $\mathrm{int} K_i \neq \varnothing$ in a control set D. Then $h_{inv}(K_1, Q) = h_{inv}(K_2, Q)$.

- For **linear control systems** in $\mathbb{R}^d$

$$x_{n+1} = Ax_n + Bu_n, u_n \in \Omega \subset \mathbb{R}^m,$$

with $\mathrm{int} K \neq \varnothing$ and (A, B) controllable, A hyperbolic and Ω a compact nbhd of 0, one has for K contained in the unique control set D

$$h_{inv}(K, D) = \sum_{\lambda \in \sigma(A)} \max(0, \log |\lambda|).$$

- hyperbolicity of the control flow on $\mathcal{U} \times Q$ gives a formula in terms of Lyapunov exponents for periodic solutions

Kawan (2014),

- also for linear control systems on Lie groups

da Silva (2014)

Comments and some further results

DA SILVA AND KAWAN, DISC. CONT. DYNAM. SYST. (2016)

Theorem. Consider a uniformly hyperbolic chain control set E with nonempty interior of a control-affine continuous time system. Assume that
(i) the Lie Algebra Rank Condition holds on $\mathrm{int}E$ and
(ii) for each $u \in \mathcal{U}$ there exists a unique $x \in E$ with $(u, x) \in \mathcal{E}$, i.e., $\mathcal{E}$ is a graph over $\mathcal{U}$.
Then E is the closure of a control set D and for every compact set $K \subset D$ with positive volume,

$$h_{inv}(K, D) = \inf_{(u,x)\in\mathcal{E}} \limsup_{t\to\infty} \log J^{+}\varphi_{t,u}(x)$$

where $J^{+}\varphi_{t,u}(x)$ is the unstable determinant of $d\varphi_{t,u}(x)$.

Invariance pressure

Introduce a potential $f \in C(\Omega, \mathbb{R})$ for the control values.
Let $K \subset Q$ be compact s.t. $\forall x \in K \ \exists u \in \mathcal{U} : \varphi(\mathbb{R}_+, x, u) \subset Q$.
A set $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)-spanning set if

$$\forall x \in K \ \exists u \in \mathcal{S} : \varphi([0, \tau], x, , u) \subset Q.$$

With $(S_\tau f)(u) := \int_0^\tau f(u(t))dt$ let

$$a_\tau(f, K, Q) := \inf\{\sum_{u \in \mathcal{S}} e^{(S_\tau f)(u)};\ \mathcal{S} \text{ is } (\tau, K, Q)\text{-spanning}\}.$$

The **invariance pressure** is

$$P_{inv}(f, K, Q) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q).$$

If $f \equiv 0$, $\sum_{u \in \mathcal{S}} e^{(S_\tau f)(u)} = \#\mathcal{S}$. Then this reduces to a known characterization of the invariance entropy.

Invariance pressure for linear control systems

Consider a **linear control systems** in $\mathbb{R}^d$

$$\dot{x} = Ax + Bu, u(t) \in \Omega \subset \mathbb{R}^m,$$

with a compact neighborhood Ω of 0 and assume (A, B) controllable, A hyperbolic.
For $K \subset D$, the unique control set with $\mathrm{int} D \neq \varnothing$, one has:

$$P_{inv}(f, K, D) \leq \sum_{\lambda \in \sigma(A)} \max(0, \mathrm{Re}\,\lambda) + \inf_{T, u(\cdot)} \frac{1}{T} \int_0^T f(u(s)) ds,$$

where the infimum is taken over all $T > 0$ and all T-periodic controls $u(\cdot)$ with values in a compact subset of $\mathrm{int}\Omega$ and a T-periodic $x(\cdot) \subset \mathrm{int} D$.

Invariance Entropy and Controllability Properties

For **dynamical systems** it is well known that the entropy is already determined on the recurrent set.

What about invariance entropy?

For **control systems** recurrence properties are replaced by controllability properties.

Here subsets of complete approximate controllability (in Q) are of relevance, called **control sets**.

W-control sets

For a **open** subset W of the state space let $\varphi_W(n,x,u)$ be the trajectories within W and define the **reachable and controllable set within W** by

$$\begin{aligned} \mathcal{R}_W(x) &= \{\varphi_W(n,x,u) \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\} \\ \mathcal{C}_W(x) &= \{y \in W \,|\, \varphi_W(n,y,u) = x \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}. \end{aligned}$$

W-control sets

For a **open** subset W of the state space let $\varphi_W(n,x,u)$ be the trajectories within W and define the **reachable and controllable set within W** by

$$\begin{aligned}\mathcal{R}_W(x) &= \{\varphi_W(n,x,u) \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\} \\ \mathcal{C}_W(x) &= \{y \in W \,|\, \varphi_W(n,y,u) = x \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}.\end{aligned}$$

Definition. A set D is called an **invariant W-control set** if
(i)

$$\overline{D}^W = \overline{\mathcal{R}_W(x)}^W \text{ for all } x \in D,$$

where the closure is taken with respect to W and
(ii) there is $x \in D$ with $x \in \mathrm{int}\mathcal{C}_W(x)$.

Remark. Condition (ii) is crucial for discrete-time systems.

Existence of invariant W-control sets

Theorem. Assume

- the state space M is a connected analytic Riemannian manifold
- $W \subset M$ is connected open and relatively compact
- the control range $\Omega \subset \overline{\mathrm{int}\Omega} \subset \mathbb{R}^m$ and $f : M \times \Omega \to M$ is analytic
- $\Omega_{sub} := \{\omega \in \Omega \,|\, f(\cdot, \omega)$ is submersive$\}$ is the complement of a proper analytic subset.

Then the following are equivalent:

(i) There are at least one and at most finitely many **invariant W-control sets** D and for every $x \in W$ there is D with

$$\mathcal{R}_W(x) \cap D \neq \emptyset.$$

(ii) There is a compact set $F \subset W$ with

$$F \cap \overline{\mathcal{R}_W(x)} \neq \emptyset \text{ for all } x \in W.$$

Albertini and Sontag (1993), Wirth (1998), Patrão and San Martin (2007)

Invariance entropy and W-control sets

Theorem. Under the assumptions of (i) in the previous theorem let $Q := \overline{W} \subset M$ and consider a compact $K \subset Q$. Assume

(i) for every relatively invariant W-control set C_i there is a compact $K_i \subset K \cap C_i$ with $\mathrm{int} K_i \neq \varnothing$.
(i) for the finitely many invariant W-control sets D_i

$$f(\bigcup_i \overline{D_i}, \Omega) \cap (\partial Q \setminus \bigcup_i \overline{D_i}) = \varnothing.$$

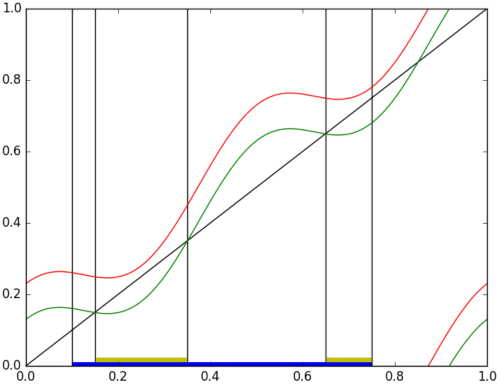
Then

$$h_{inv}(K, Q) = \max_i h_{inv}(K_i, C_i).$$

where the maximum is taken over all relatively invariant W-control sets C_i.

Remark. In the continuous-time case a similar result has been shown in FC/Lettau (2016).

$$f_\alpha(x,\omega) = x + \sigma\cos(4\pi x) + A\omega + \alpha \mod 1.$$



Two W-control sets D_1 and D_2 (to the right) in $W = (0.1, 0.7)$. The invariance entropies on $Q = [0.1, 0.7]$ and on $\overline{D_2}$ coincide.

Final remarks

Classical entropy of dynamical systems describes the **total information** generated by the system topologically or with respect to an **invariant measure**.

In contrast, entropy for control systems describes the **minimal information** for invariance in a topological context.

The data rate theorem relates the topological invariance entropy to the minimal bit rate needed for invariance.

In a similar vein, minimal bit rates for other control problems, e.g. stabilization or state estimation, can be determined.

There are also several efforts to develop a measure-theoretic notion of invariance entropy.

Some references

F. Colonius, J. Cossich, A. Santana, Invariance pressure of control sets, SIAM J. Control Optim. 56 (2018), pp. 4130-4147.

F. Colonius, Invariance entropy, quasi-stationary measures and control sets. Discrete and Continuous Dynamical Systems (DCDS-A) 38(4) (2018), 2093-2123.

F. Colonius, R. Lettau, Relative controllability properties, IMA Journal of Mathematical Control and Information 33 (2016), 701-722.

A. da Silva and C. Kawan, Invariance entropy of hyperbolic control sets, Discrete and Continuous Dynamical Systems (DCDS-A), 36 (2016), 97-136.

G. Nair, R.J. Evans, I. Mareels, and W. Moran, Topological feedback entropy and nonlinear stabilization, IEEE Transactions on Automatic Control 49 (2004), 1585-1597.

Xingfu Zhong and Yu Huang, Invariance Pressure Dimensions for Control Systems, Journal of Dynamics and Differential Equations (2018).